Oliver's Notes in Mathematics and Physics

2. Introductory Classical Mechanics

Newtonian and Lagrangian Mechanics

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2. Introductory Classical Mechanics

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Preface

This text serves as a first rigorous introduction to the study of classical mechanics in physics. A high school course in physics and calculus should be sufficient to understand the content of this book.

The book begins with kinematics, and then begins to focus on Newton's laws. After that, certain topics such as work, energy, momentum, and impulse are considered. The text then shifts into Lagrangian mechanics, and concludes with a vast array of applications ranging from waves to planetary orbits.

I would like to thank David Morin, Lev Landau, and Evgeny Lifshitz for writing some of the texts that have inspired this one. David Morin's *Introduction to Classical Mechanics* is the quintessential honors freshman physics textbook. Lev Landau and Evgeny Lifshitz's classical mechanics text will serve to supplement many of the topics in this collection of notes. I would also like to thank my physics professor, Professor Katerina Visnjic, as well as my preceptor, Professor Simone Giombi.

How to read this book

Definitions are highlighted in blush boxes; examples appear in pale lavender; and brief conceptual remarks use a quieter neutral frame. Important results are set as theorems or propositions, followed by proofs in the main text. Exercises at the end of each chapter are intended to be part of the exposition rather than an afterthought; the reader should complete all of the exercises, as there are not many of them. There are also problems scattered throughout each chapter; the reader should also complete all of those in order to gain the most complete understanding of the subject. Problems are generally easier and can even be done in the reader's head; exercises are generally harder and require more thought.

It is possible, and recommended, to fit the entire content of this book into one semester, with some room for additional topics. Alternatively, this book could be used as the basis for a very leisurely two semester course, alongside supplementary texts provided by the professor as necessary.

Oliver Kahng
2026

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CHAPTER 1

Kinematics

"Study hard what interests you the most in the most undisciplined, irreverent and original manner possible."
Richard Feynman

1.1 What is Physics?

What comes to mind when you think of physics? Perhaps a ball being thrown in the air, or perhaps a rocket being launched. Perhaps you think of the smallest forces between atoms, or perhaps you think of the stars above. Physics can be adequately summed up as the following: physics is the science of using mathematical models to predict the future.

In physics, we are often given **initial conditions**. For example, we might know the velocity of an object at a certain time. Or we may know the mass of the object. We are also given information about the environment. What forces are acting on such an object? What other objects are there? How does the whole system we are considering come together and connect to each other? We use **physical tools** in order to obtain a prediction of the phenomena that we are considering.

Oftentimes, physicists do not always have the tools readily defined, available for use. Such tools necessarily to analyze the phenomena that are being considered must sometimes be developed as well. It is of utmost importance that a budding physicist understands this concept and has a taste for what it feels like to develop physical tools and mathematical models from scratch.

1.2 Kinematics in Cartesian Coordinates

Kinematics is defined as the precise mathematical description of motion. The reader is certainly familiar with the notion of kinematics in Cartesian coordinates. In order to discuss this, however, we must fix an origin and a coordinate system in order to unambiguously describe the system. That is, we define a certain point to be the origin and define certain directions to show where the x- and y-axes point.

We consider what is known as the **position vector**, which is simply just denoted $\vec{r}$, and is equal to

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \tag{1.1}$$

Note that x, y, z are the *components* of the vector. It just so happens that they are the same as the *coordinates* of the vector (that is, $\vec{r}$ can be represented as an ordered triple (x, y, z) based on the

coordinate axes. The components of the vector are essentially the magnitudes of the vectors along the x-, y-, and z-axes such that the sum of all such vectors equals $\vec{r}$. Now, what are $\hat{i}$, $\hat{j}$, and $\hat{k}$? These are the **unit vectors**, which are vectors with magnitude 1. These vectors have *no units*. They essentially serve as a means to point in a given direction. $\hat{i}$ points in the direction toward the positive x-axis, and it is similar with $\hat{j}$ and the y-axis, and $\hat{k}$ and the z-axis.

It is well-known to the reader that we can define the **velocity vector** as the time derivative of the position vector, like so:

$$\vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dt} \tag{1.2}$$

The reader may be familiar with the use of the "prime" $'$ to denote the derivative; for example, $f'(x)$ denotes df/dx. However, in physics, the prime symbol is most often used to denote the derivative with respect to position; the derivative with respect to time is denoted using the dot, like so: $\dot{x} = dx/dt$. It necessarily follows that we have that

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} \tag{1.3}$$

as is derived in a typical high school physics textbook.

Similarly, we have that the **acceleration vector** is simply the second derivative of the position vector with respect to time:

$$\vec{a} = \ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} \tag{1.4}$$

There exist names for higher derivatives of the position vector. For example, the third derivative $\dddot{\vec{r}}$ is called the **jerk**. The fourth, fifth, and sixth derivatives are called the **snap**, **crackle**, and **pop**, respectively. Jerk is sometimes important in engineering applications; otherwise, there are not many uses for these quantities.

Notice that we are able to decompose our velocity vector into components with respect to the unit vectors, despite the velocity vector originating at a point that is not necessarily the origin. Why is that so? Every point in space can be thought of as having a "frame" of unit vectors, which we are able to decompose the velocity vector into. One can think of such a frame consisting of the vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ as traveling *with* a particle as it moves, so that we may be able to decompose the velocity vector into such unit vectors.

Suppose that a particle traveling through space has constant acceleration. You are undoubtedly familiar with this; if not, you will recognize the result soon enough. So, let $a = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$, where a_x, a_y, a_z are constant. We are able to recover the velocity by integrating; for example, the x-velocity can be obtained by integrating the x-acceleration:

$$v_x(t) = \int_0^t a_x dt' = a_x t + v_{0x} \tag{1.5}$$

There are a couple of things to note here. Notice that we labeled the constant term v_{0x}. This is called the **initial velocity**, and it is the velocity of the particle at time $t = 0$. Additionally, we are unable to use t as both the final variable and the variable being integrated over. So instead, we employ a **dummy variable**, which we denote t' (yes, that is not a typo). Now, we may recover the x coordinate of the particle by integrating the x-velocity:

$$x(t) = \int_0^t v_x(t')dt' = \frac{1}{2}a_x t^2 + v_{0x}t + x_0 \tag{1.6}$$

Our constant term here, x_0, is called the **initial position**, and represents the x coordinate of the particle at time $t = 0$. We have derived, through integration, one of the fundamental kinematic equations.

1.3 Kinematics in Polar Coordinates

Now it is time for things to get a little bit more interesting. In 3 dimensions, instead of Cartesian coordinates, we may employ the use of spherical or cylindrical coordinates. However, we will limit ourselves to 2 dimensions, in which we will use **polar coordinates**, which define position based on the distance from the origin as well as the angle it makes with a given axis. We are able to represent x and y coordinates in terms of polar coordinates like so:

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1.7}$$

How are we able to find the unit vectors at any given point? To do so, fix almost all of the coordinates, except the one we are wishing to find the unit vector for, and increase the value of the coordinate; the unit vector we are looking for should be tangent to the path traced out by increasing the coordinate.

How would we represent the position vector in polar coordinates? To do so, we will consider the unit vectors at the tip of the position vector. Since the unit r vector points outward, and the unit θ vector points perpendicular to the unit r vector (and $\vec{r}$ itself), we have that

$$\vec{r} = r\hat{r} + 0\hat{\theta} = r\hat{r} \tag{1.8}$$

So, we are looking at the vector pointing in the direction of $\hat{r}$ that is of length r; that is the position vector in polar coordinates.

What if it was that $\vec{r} = r\hat{r} + \theta\hat{\theta}$ instead? We are able to tell immediately that this is not the case, for the units do not match. Recall that the unit vectors have *no* units. r has units of meters, and θ has units of radians. The units do not match. So, let us use Equation 1.8 instead of whatever this is.

Equation 1.8 highlights a very important distinction: coordinates are not the same as components. The coordinates of $\vec{r}$ are found in the ordered pair (r, θ), yet the components of $\vec{r}$ based on the unit vectors are r and 0. Why do they seem to be equivalent in Cartesian coordinates, then?

Notice that, thinking back to Cartesian coordinates, the unit vectors $\hat{i}, \hat{j}, \hat{k}$ do not change as a particle moves around; however, $\hat{r}$ and $\hat{\theta}$ change. For example, $\hat{r}$ always must point *away* from the origin, and so it may vary.

If we wish to find the velocity vector in polar coordinates, we employ the product rule like so:

$$\vec{v} = \dot{\vec{r}} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\dot{\hat{r}} \tag{1.9}$$

Notice that if we use the product rule the same way when taking the velocity in Cartesian coordinates, the rate of change of any of the unit vectors, $\dot{\hat{i}}$ for example, is always equal to 0, so our previous derivation from Section 1.2 holds. However, $\dot{\hat{r}}$ is not necessarily 0, so we must be more careful here.

This begs the question, what is $\dot{\hat{r}}$? In order to find out, we must find $\hat{r}$ in terms of Cartesian coordinates. Trigonometry allows us to write

$$\hat{r} = \cos(\theta)\hat{i} + \sin(\theta)\hat{j} \tag{1.10}$$

and

$$\hat{\theta} = -\sin(\theta)\hat{i} + \cos(\theta)\hat{j} \tag{1.11}$$

We may also represent $\hat{i}$ and $\hat{j}$ in terms of $\hat{r}$ and $\hat{\theta}$ by inverting the associated matrix:

$$\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{i}\\ \hat{j}\end{pmatrix} \implies \begin{pmatrix}\hat{i}\\ \hat{j}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix} \tag{1.12}$$

$$\hat{i} = \cos(\theta)\hat{r} - \sin(\theta)\hat{\theta} \tag{1.13}$$

$$\hat{j} = \sin(\theta)\hat{r} + \cos(\theta)\hat{\theta} \tag{1.14}$$

The rightmost matrix is known as a **rotation matrix** by the angle θ (technically, the left matrix is also a rotation matrix by the angle $-\theta$, but the standard form of the rotation matrix is the right matrix). Therefore, we may interpret the polar frame as being the Cartesian frame rotated by θ.

Problem 1.1

Recall that I stated earlier that $\hat{r}$ and $\hat{\theta}$ are perpendicular. Using the representation of these vectors in Cartesian coordinates, verify this statement.

Notice that $\hat{r}$ depends on θ, not t (time). However, it may be true that θ depends on t, so it is said that $\hat{r}$ is **implicitly dependent** on t. Therefore, using the multivariable chain rule, we obtain

$$\dot{\hat{r}} = \frac{d\hat{r}}{dt} = \frac{\partial\hat{r}}{\partial\theta}\cdot\frac{d\theta}{dt} \tag{1.15}$$

Using our decomposition of $\hat{r}$ into Cartesian unit vectors, we obtain that the above expression is equal to

$$-\sin(\theta)\dot{\theta}\hat{i} + \cos(\theta)\dot{\theta}\hat{j} = \dot{\theta}(-\sin(\theta)\hat{i} + \cos(\theta)\hat{j}) = \dot{\theta}\hat{\theta} \tag{1.16}$$

It is wise to check our units to make sure that $\dot{\hat{r}} = \dot{\theta}\hat{\theta}$. We have that $\dot{\hat{r}}$ is in units of s^{-1} (recall that $\hat{r}$ is unitless, so the time derivative adds s^{-1}). We also have that $\dot{\theta}$ is in s^{-1} and $\hat{\theta}$, of course, has no units. Everything looks good.

Problem 1.2

Another result that we will need is the time derivative of $\hat{\theta}$. Show that

$$\dot{\hat{\theta}} = -\dot{\theta}\hat{r} \tag{1.17}$$

Now, using Equation 1.9, we are able to calculate $\vec{v}$ by substituting in our expression for $\dot{\hat{r}}$, like so:

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{1.18}$$

It is time to analyze the differing terms in this equation. The **radial component of velocity**, v_R, is the component associated with the unit vector $\hat{r}$. This component is simply $\dot{r}$. The term $\dot{\theta}$ should

be a little bit familiar; it is what is known as the **angular velocity**. This quantity, multiplied by the radius r, gives the **tangential velocity** v_T, which is equal to $r\dot{\theta}$, and is the component of $\vec{v}$ associated with the unit vector $\hat{\theta}$.

We now consider the acceleration, which is left to the reader:

Problem 1.3

Show that

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} \tag{1.19}$$

Find this quantity by taking the derivative of the velocity vector $\vec{v}$ in polar coordinates and applying appropriate rules from calculus.

Certain terms are of interest here. The **radial component of acceleration** is the component associated with $\hat{r}$, which is, of course, $\ddot{r} - r\dot{\theta}^2$. The **tangential component of acceleration** is the component associated with $\hat{\theta}$, which is $2\dot{r}\dot{\theta} + r\ddot{\theta}$.

The quantity $\ddot{r}$ can be thought of as the radial contribution to the radial component of acceleration. The other term in the radial component, $-r\dot{\theta}^2$, is known as the **centripetal acceleration**; it is an acceleration pointing inward when there is a nonvanishing angular velocity. The **angular acceleration** is the derivative of the angular velocity, $\ddot{\theta}$. When multiplied by r, so that we have $r\ddot{\theta}$, we obtain the "tangential acceleration" familiar to the reader from high school. Truthfully, this is only the tangential acceleration if the radius remains constant. Otherwise, we must consider another term, namely $2\dot{r}\dot{\theta}$, which is called the **Coriolis term**.

We may be concerned with the magnitude squared of velocity. Since the polar frame is orthonormal, we may compute the square of the magnitude like so:

$$\vec{v}^2 = \vec{v} \cdot \vec{v} = (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \cdot (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \tag{1.20}$$

$$= \dot{r}^2(\hat{r} \cdot \hat{r}) + 2r\dot{r}\dot{\theta}(\hat{r} \cdot \hat{\theta}) + r^2\dot{\theta}^2(\hat{\theta} \cdot \hat{\theta}) = \dot{r}^2(1) + 2r\dot{r}\dot{\theta}(0) + r^2\dot{\theta}^2(1) = \dot{r}^2 + r^2\dot{\theta}^2 \tag{1.21}$$

We may derive this expression in another way. We understand that the square of the magnitude of the velocity in Cartesian coordinates is

$$\vec{v}^2 = \dot{x}^2 + \dot{y}^2 \tag{1.22}$$

Take the derivatives of $x = r\cos\theta$ and $y = r\sin\theta$ with respect to time:

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \tag{1.23}$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \tag{1.24}$$

Then, we may compute $\vec{v}^2$ accordingly:

$$\vec{v}^2 = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2 = \dot{r}^2 + r^2\dot{\theta}^2 \tag{1.25}$$

It is sometimes easier to work with scalar quantities such as the magnitude squared of the velocity rather than vectors such as the velocity itself. The obvious application of such is energy, specifically kinetic energy, but when using Lagrangians (which themselves are quite related to energy), scalar quantities such as the square of the magnitude of the velocity are of great use.

1.4 Examples and Applications

There are a wide variety of examples that we may begin to appreciate. We consider a few below:

Example: Uniform Circular Motion

Impose a polar coordinate system on a plane. Suppose that we have a particle traveling at a fixed distance R around the origin, with fixed angular velocity ω. We find that, strictly in terms of coordinates of the particle, that

$$r(t) = R, \quad \theta(t) = \int_0^t \omega dt' = \omega t \tag{1.26}$$

This implies that $\dot{\theta} = \omega$ where ω is constant, as previously stated, and furthermore, $\ddot{\theta} = 0$. Our position vector, as mentioned before, is $\vec{r}(t) = R\hat{r}$. Taking the derivative of the position, we may obtain the velocity vector:

$$\vec{v}(t) = \dot{\vec{r}} = R\dot{\hat{r}} = R\omega\hat{\theta} \tag{1.27}$$

Once more taking the derivative gives us the acceleration vector, like so:

$$\vec{a}(t) = \dot{\vec{v}}(t) = R\omega\dot{\hat{\theta}} = -R\omega^2\hat{r} \tag{1.28}$$

We make a couple of observations. First, we see that

$$\vec{v}^2 = R^2\omega^2 \tag{1.29}$$

which is constant. Therefore, taking the derivative of this scalar quantity should give us zero. So, we have that

$$\frac{d}{dt}\vec{v}^2 = \frac{d}{dt}(\vec{v}\cdot\vec{v}) = 0 \tag{1.30}$$

$$\implies 2(\vec{a}\cdot\vec{v}) = 0 \tag{1.31}$$

so it is clear that $\vec{a}$ is orthogonal to $\vec{v}$ in this example.

Is the acceleration of a particle always orthogonal to the velocity when the particle undergoes circular motion? Not so, as the following example illustrates.

Example: Non-Uniform Circular Motion

Impose a polar coordinate system on a plane. Suppose that we have a particle that travels at a fixed distance R from the origin, yet its angular jerk is a constant κ. It necessarily follows that we may represent this particle's position in terms of polar coordinates like so:

$$r(t) = R, \quad \theta(t) = \frac{\kappa}{6}t^3 \tag{1.32}$$

since $\dddot{\theta} = \kappa$, which is constant. Recall once more that the position vector may be represented as $\vec{r}(t) = R\hat{r}$. Then, we have that

$$\vec{v}(t) = R\dot{\theta}\hat{\theta} = \frac{R\kappa t^2}{2}\hat{\theta} \tag{1.33}$$

Notice that the velocity is substantially less trivial than in the example before. Calculating acceleration, we must use the product rule to obtain the following.

$$\vec{a}(t) = R\ddot{\theta}\hat{\theta} - R\dot{\theta}^2\hat{r} = -\frac{R\kappa^2 t^4}{4}\hat{r} + R\kappa t\hat{\theta} \tag{1.34}$$

It is clear at this point that $\vec{a}$ is not orthogonal to $\vec{v}$, as $\vec{v}$ is only decomposed into $\hat{\theta}$, whereas $\vec{a}$ is not decomposed into only $\hat{r}$, which is what one would expect for an acceleration orthogonal to velocity.

Another example of interest occurs regarding a particle whose radial position is not necessarily constant. Observe the following example below:

Example: Ant on a Rotating Disk

Consider a disk rotating counterclockwise at an angular velocity ω. Impose a polar coordinate system such that the origin is at the center of the disk and at $t = 0$, we have $\theta(t) = 0$. An ant on the disk moves radially outward from the center with a constant radial velocity v_0 from the reference frame of the disk. Notice that the spinning disk has no impact on the radial velocity of the ant, so in the observer's frame, the ant has radial velocity v_0, as well as angular velocity ω due to being present on the spinning disk. Therefore, its motion can be described using the parametric equations

$$r(t) = v_0 t, \quad \theta(t) = \omega t \tag{1.35}$$

It is worthwhile to mention that in this case, r may be expressed in terms of θ like so: $r(\theta) = \frac{v_0}{\omega}\theta$. We have that

$$\vec{r} = r\hat{r} = v_0 t\hat{r} \tag{1.36}$$

which implies

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = v_0\hat{r} + v_0\omega t\hat{\theta} \tag{1.37}$$

and

$$\vec{a} = v_0\omega\hat{\theta} + v_0\omega\hat{\theta} - v_0\omega^2 t\hat{r} = -v_0\omega^2 t\hat{r} + 2v_0\omega\hat{\theta} \tag{1.38}$$

It is an interesting shape, the shape that the ant traces out in this example. It is a curve known as an **Archimedean spiral**.

To conclude this chapter, a couple of tips:

Remark.

When discussing problems, you may want to consider all the forces (or lack thereof) that are acting on a given object. Note that a force implies acceleration. For example, if you are given that a particle is frictionless and moves along the spoke of a rotating wheel, consider the number of forces operating radially on the particle. There are none. Therefore, you may be able to set the radial component of acceleration to zero. Likewise, if there is an acceleration somewhere which changes a particle's velocity, then there is some force (which may be "hidden") acting on the particle.

Remark.

Let us discuss **differential equations** for a moment. Suppose that we have a differential equation of the form

$$\ddot{y} + a\dot{y} + by = 0 \tag{1.39}$$

Let us "try" the solution e^{kx} for some k:

$$k^2e^{kx} + ake^{kx} + be^{kx} = 0 \implies (k^2 + ak + b)e^{kx} = 0 \tag{1.40}$$

Now, e^{kx} is never 0, so that must mean $k^2 + ak + b = 0$. We may use the quadratic formula to solve this equation:

$$k = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \tag{1.41}$$

So we have

$$y = C_1e^{k_1x} + C_2e^{k_2x} \tag{1.42}$$

where k_1, k_2 are the solutions k to Equation 1.41, and we have arbitrary constants C_1, C_2. Some special solutions of note: if there is only one such solution k to Equation 1.41, then our solution is instead

$$y = (C_1 + C_2x)e^{kx} \tag{1.43}$$

Finally, if our solutions k_1, k_2 are both complex and can be expressed in the forms $a + bi, a - bi$ (note that they are complex conjugates), then we may write

$$y = e^{ax}(C_1\cos(bx) + C_2\sin(bx)) \tag{1.44}$$

Remark.

Sometimes, performing **dimensional analysis** on quantities allows for key insights into whether a solution is correct or not. To begin, consider a quantity, however complicated it may be, and consider the desired units. Then, take the units of each part of the quantity, and perform necessary cancellations and simplifications (this should be review from high school). Something may be wrong; for example, two quantities added together may have different units, or the final unit is unexpected. This reveals that somewhere, things have gone wrong. Or perhaps your thought process was incorrect. Something has happened, and it would behoove you to rectify that.

Remark.

When solutions are much too complicated to discuss effectively, for certain appropriate values, one may want to consider the first couple terms of a **Taylor series** of the function, which is essentially a polynomial that approximates a function at certain values. For example, we may use the Taylor expansion $\sqrt{1 + x^2} \approx 1 + x/2$ for very small values of x, such as $x \ll 1$.

1.5 Exercises

Exercises

1. Suppose $\hat{a}$ and $\hat{b}$ are unit vectors in the Cartesian plane making angles θ, ϕ with the x-axis.
   (a) Find $\hat{a}$ and $\hat{b}$ in terms of the standard unit vectors $\hat{\imath}$ and $\hat{\jmath}$. Additionally, prove the identity $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$ using dot products.
   (b) Show that the matrix
   $$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{1.45}$$
   rotates the vector $\hat{a}$ by the angle α.
   (c) What is the resulting matrix when you compose the matrix of a rotation α and the matrix of a rotation β?
2. Consider a particle moving in the xy-plane, with position described by functions $x(t)$ and $y(t)$.
   (a) Express the x-velocity of the particle at $t = 1$ s.
   (b) How would one find the position of the particle the first time it is at rest (perhaps instantaneously) in the x-direction?
   (c) Find the acceleration in the x-direction of the particle at the time in part (a).
   (d) How would one find the velocity of the particle the first time it passes through the y-axis?
3. We now work with frames of reference O and O' in a single dimension, where O' is moving with speed u relative to O. Suppose that at $t = 0$, the systems coincide. At some time t, O' has moved by a distance $s = ut$, and it is not hard to deduce that the position x_1' of a particle in the frame O', given x_1 is the position in the frame O, is $x_1' = x_1 - s = x_1 - ut$. Differentiating this, we obtain the velocity in the frame O' relative to O: $v' = v - u$. Note that we are working in Newtonian physics.
   (a) Find the acceleration of the particle in the frame O' in terms of the acceleration in the frame O.
   (b) What happens if O and O' coincide at $t = 0$, but O' moves with constant acceleration relative to O? Find s as defined before.
   (c) Suppose the position, velocity, and acceleration of a particle in O are x_2, v_2, and a_2. What is x_2', v_2', a_2' in the frame O'?
4. (a) How does the speed of waves on a string depend on its mass M, length L, and tension (that is, force) T?
   (b) A ball is thrown at an angle θ up to the top of a cliff of height L, from a point a distance L from the base, as shown in Fig. 1.9. Assuming that one of the following quantities is the initial speed required to make the ball hit right at the edge of the cliff, which one is it? (Don't solve the problem from scratch, just check special cases.)
   $$\sqrt{\frac{gL}{2(\tan\theta - 1)}}, \frac{1}{\cos\theta}\sqrt{\frac{gL}{2(\tan\theta - 1)}}, \frac{1}{\cos\theta}\sqrt{\frac{gL}{2(\tan\theta + 1)}}, \sqrt{\frac{gL\tan\theta}{2(\tan\theta + 1)}} \tag{1.46}$$
5. Suppose that a particle is moving freely along the spoke of a wheel, which is rotating at constant angular speed ω. The particle is frictionless. $r(0) = r_0$ and $v_r(0) = 0$. Find

$r(t)$.

6. A stick with length 10 cm is sliding on a frictionless surface toward a wall. At $t = 0$, the stick is 120 cm away from the wall and the center moves toward the wall with a speed $v = 20$ cm/s. The stick is also spinning (either clockwise or counterclockwise) at a constant rate, ω rad/s. Let $\theta(t) = \omega t$, and $\theta = 0$ implies that the stick is parallel to the wall. Find all values of ω such that the stick hits the wall in such a way that it is parallel to the wall. It is not allowed to phase through the wall at any time.

CHAPTER 2

Newton's Laws

Chapter 3

Oscillations

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Work and Energy

Chapter 5

Impulse and Momentum

Chapter 6

Lagrangian Formalism

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Planetary Motion

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Angular Momentum

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Waves

Further Reading

- Morin, *Introduction to Classical Mechanics*.
- Landau and Lifshitz, *Classical Mechanics*.